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Statistics of first-passage times in disordered systems using backward master equations and their exact renormalization rules

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Abstract

We consider the non-equilibrium dynamics of disordered systems as defined by a master equation involving transition rates between configurations (detailed balance is not assumed). To compute the important dynamical time scales in finite-size systems without simulating the actual time evolution which can be extremely slow, we propose to focus on first-passage times that satisfy 'backward master equations'. Upon the iterative elimination of configurations, we obtain the exact renormalization rules that can be followed numerically. To test this approach, we study the statistics of some first-passage times for two disordered models: (i) for the random walk in a two-dimensional self-affine random potential of Hurst exponent H, we focus on the first exit time from a square of size $L \times L$ if one starts at the square center and (ii) for the dynamics of the ferromagnetic Sherrington–Kirkpatrick model of N spins, we consider the first passage time t_f to zero-magnetization when starting from a fully magnetized configuration. Besides the expected linear growth of the averaged barrier $\overline{\ln t_f} \sim N$, we find that the rescaled distribution of the barrier $(\ln t_f)$ decays as e^{-u^η} for large u with a tail exponent of order $\eta \simeq 1.72$. This value can be simply interpreted in terms of rare events if the sample-to-sample fluctuation exponent for the barrier is $\psi_{\text{width}} = 1/3$.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In statistical physics, any large-scale universal behavior is expected to come from some underlying renormalization 'RG' procedure that eliminates all the details of microscopic models. For the non-equilibrium dynamics of disordered systems, we have recently proposed a strong disorder renormalization procedure in configuration space that can be defined for any

master equation [1–3]: it is based on the iterative elimination of the smallest barrier remaining in the system, and thus generalizes the real-space strong disorder procedures that had been previously defined for random walks in one-dimensional random media [4–8]. However, as for all strong disorder renormalization procedures (see [9] for a review), the results are asymptotically exact only near 'infinite disorder fixed points': for the dynamical problems defined by a master equation, this means that the strong disorder renormalization procedure will give asymptotically exact results only if the renormalized distribution of barriers becomes broader and broader upon iteration (see [1] for a more detailed discussion). In the present paper, we show that one can obtain exact renormalization rules, *without any strong disorder hypothesis,* if one considers the 'backward master equation' satisfied by first-passage times. It turns out that the renormalization rules for the transition rates are formally identical to the strong disorder rules introduced in [1, 2], but the interpretation, the goals and the validity of the two approaches are different, as we explain in more details below.

From a numerical point of view, the main limitation of Monte Carlo dynamical simulations of disordered systems is that the dynamics in the presence of quenched disorder becomes extremely slow as the system size increases (see for instance the introduction of our recent work [10] and references therein). It is thus important to develop other methods to characterize the dynamical properties of disordered systems *without simulating the dynamics*. For instance in our previous work [10], we have proposed to use the mapping between any master equation satisfying detailed balance and some Schrödinger equation in configuration space, to obtain the largest relaxation time of the dynamics via any eigenvalue method able to compute the energy of the first excited state of the associated quantum Hamiltonian. Here we propose another strategy based on the 'backward master equation' satisfied by first-passage times. The fact that first-passage times satisfy 'backward master equation' is of course very well known and can be found in most textbooks on stochastic processes (see for instance [11–14]). In the field of disordered systems, the backward Fokker–Planck equation has been very much used to characterize the dynamics of a single particle in a random medium (see for instance [5, 15–20]), but to the best of our knowledge, this approach has not yet been used in higher dimensions, nor for many-body problems. To test the present approach, we compute the statistics of first-passage times over the disordered samples of a given size for two disordered models: (i) a random walk in a two-dimensional random potential and (ii) a mean-field spin model.

The paper is organized as follows. In section 2, we recall that first-passage times satisfy the 'backward master equation'. In section 3, we derive the corresponding renormalization rules and discuss the similarities and differences with respect to strong disorder renormalization procedures. We then apply this approach to two types of disordered models: section 4 concerns the problem of a random walk in a two-dimensional self-affine potential, and section 5 is devoted to the dynamics of the ferromagnetic Sherrington–Kirkpatrick (SK) model. Our conclusions are summarized in section 6.

2. Reminder on first-passage times and backward master equations

2.1. Master equation defining the stochastic dynamics

In statistical physics, it is convenient to consider continuous-time stochastic dynamics defined by a 'forward' master equation of the form

$$\frac{\mathrm{d}P_t(\mathcal{C})}{\mathrm{d}t} = \sum_{\mathcal{C}'} P_t(\mathcal{C}')W(\mathcal{C}' \to \mathcal{C}) - P_t(\mathcal{C})W_{\mathrm{out}}(\mathcal{C}) \tag{1}$$

that describes the evolution of the probability $P_t(\mathcal{C})$ to be in configuration $\mathcal{C}$ at time t. The notation $W(\mathcal{C}' \to \mathcal{C})$ represents the transition rate per unit time from configuration $\mathcal{C}'$ to $\mathcal{C}$, and

$$W_{\rm out}(\mathcal{C}) \equiv \sum_{\mathcal{C}'} W(\mathcal{C} \to \mathcal{C}') \tag{2}$$

represents the total exit rate out of configuration $\mathcal{C}$.

2.2. Backward master equation satisfied by the first-passage time

Let us now focus on the following problem: suppose the dynamics starts at $t = 0$ in configuration $\mathcal{C}$, and one is interested in the random time t where the dynamics will reach for the first time any configuration belonging to a given set A of 'target' configurations. As is well known (see for instance the textbooks [11–14]), the mean first-passage time $\tau^{(A)}(\mathcal{C}) = \langle t \rangle$ (where the notation $\langle . \rangle$ represents the average with respect to the dynamical trajectories) satisfies the following 'backward master equation' for all configurations $\mathcal{C}$ not in the set A:

$$\sum_{\mathcal{C}'} W(\mathcal{C} \to \mathcal{C}')\tau^{(A)}(\mathcal{C}') - W_{\rm out}(\mathcal{C})\tau^{(A)}(\mathcal{C}) = -1, \tag{3}$$

whereas all configurations in the set A satisfy the boundary conditions

$$\tau^{(A)}(\mathcal{C} \in A) = 0. \tag{4}$$

The derivation of equation (3) consists in considering what happens during the first time interval $[0, {\rm d}t]$ if the system is in configuration $\mathcal{C}$ at $t = 0$: at time ${\rm d}t$, the system is either in configuration $\mathcal{C}'$ with the probability $[W(\mathcal{C} \to \mathcal{C}')\,{\rm d}t]$, in which case the remaining mean time is $\tau^{(A)}(\mathcal{C}')$, or the system is still in configuration $\mathcal{C}$ with the probability $[1 - W_{\rm out}(\mathcal{C})\,{\rm d}t]$, in which case the remaining mean time is $\tau^{(A)}(\mathcal{C}\,)$. By consistency, the mean first passage time has thus to satisfy at first order in ${\rm d}t$:

$$\tau^{(A)}(\mathcal{C}) = {\rm d}t + \sum_{\mathcal{C}'}[W(\mathcal{C} \to \mathcal{C}')\,{\rm d}t]\,\tau^{(A)}(\mathcal{C}') + [1 - W_{\rm out}(\mathcal{C})\,{\rm d}t]\tau^{(A)}(\mathcal{C}), \tag{5}$$

yielding equation (3).

The backward master equations of equation (3) can be solved numerically by any method appropriate for linear equations with fixed right-hand side. In the next section, we show that they satisfy exact renormalization rules.

3. Renormalization rules for first-passage time properties

3.1. Iterative elimination of configurations

If one eliminates iteratively the configurations from the system of equation (3) satisfied by the first-passage times, the renormalized equations for the surviving configurations keep the same form, but with renormalized transition rates W^R and renormalized right-hand sides K^R:

$$\sum_{\mathcal{C}'} W^R(\mathcal{C} \to \mathcal{C}')\tau^{(A)}(\mathcal{C}') - W^R_{\rm out}(\mathcal{C})\tau^{(A)}(\mathcal{C}) = -K^R(\mathcal{C}). \tag{6}$$

This equation for $\mathcal{C} = \mathcal{C}_0$ can be used to eliminate $\tau^{(A)}(\mathcal{C}_0)$ via

$$\tau^{(A)}(\mathcal{C}_0) = \frac{1}{W^R_{\rm out}(\mathcal{C}_0)}\left[\sum_{\mathcal{C}''} W^R(\mathcal{C}_0 \to \mathcal{C}'')\tau^{(A)}(\mathcal{C}'') + K^R(\mathcal{C}_0)\right]. \tag{7}$$

Upon the elimination of the configuration $\mathcal{C}_0$, the renormalized coefficients W^R and K^R evolve according to the following renormalization rules for the surviving configurations $\mathcal{C}$:

$$\begin{aligned} W^{R\text{new}}(\mathcal{C} \to \mathcal{C}') &= W^R(\mathcal{C} \to \mathcal{C}') + \frac{W^R(\mathcal{C} \to \mathcal{C}_0) W^R(\mathcal{C}_0 \to \mathcal{C}')}{W^R_{\text{out}}(\mathcal{C}_0)} \\ W^{R\text{new}}_{\text{out}}(\mathcal{C}) &= W^R_{\text{out}}(\mathcal{C}) - \frac{W^R(\mathcal{C} \to \mathcal{C}_0) W^R(\mathcal{C}_0 \to \mathcal{C})}{W^R_{\text{out}}(\mathcal{C}_0)} \\ K^{R\text{new}}(\mathcal{C}) &= K^R(\mathcal{C}) + \frac{W^R(\mathcal{C} \to \mathcal{C}_0)}{W^R_{\text{out}}(\mathcal{C}_0)} K^R(\mathcal{C}_0). \end{aligned} \tag{8}$$

3.2. Renormalization rules for other observables satisfying 'backward master equation'

Since other observables are known to satisfy similar 'backward master equations', it is interesting to discuss here their renormalization rules and to compare with equation (8).

3.2.1. Higher moments of first-passage times. Above we have considered the first moment $\tau^{(A)}(\mathcal{C}) = \langle t \rangle$ of the first-passage time in the set A when starting in configuration $\mathcal{C}$. However, one may consider the higher moments $\tau_n^{(A)}(\mathcal{C}) = \langle t^n \rangle$ that satisfy the following 'backward master equation' [11–14]) for all configurations $\mathcal{C}$ not in the set A:

$$\sum_{\mathcal{C}'} W(\mathcal{C} \to \mathcal{C}') \tau_n(\mathcal{C}') - W_{\text{out}}(\mathcal{C}) \tau_n(\mathcal{C}) = -n \tau_{n-1}(\mathcal{C}) \tag{9}$$

whereas all configurations in the set A satisfy the boundary conditions

$$\tau_n(\mathcal{C} \in A) = 0. \tag{10}$$

The derivation of equation (9) consists again in considering what happens during the first time interval $[0, \mathrm{d}t]$ (see explanations before equation (5)). The higher moments of first-passage times can be thus computed one after the other: if one knows the moments of order $(n-1)$, one can compute the moments of order n via the same renormalization rules of equation (8): the only change will be in the initial condition for the right-hand side that will read $K_n^{\text{initial}}(\mathcal{C}) = n \tau_{n-1}(\mathcal{C})$ instead of $K_{n=1}^{\text{initial}}(\mathcal{C}) = 1$.

3.2.2. Escape probabilities. The simplest quantities that satisfy some backward master equation are the escape probabilities. Suppose the dynamics starts at $t = 0$ in configuration $\mathcal{C}$, and one is interested in the probability $E_{B/A}(\mathcal{C})$ to reach first any configuration belonging to a set B of configurations before any configuration belonging to another set A of configurations. As is well known (see for instance the textbooks [11–14]), this escape probability $E_{B/A}(\mathcal{C})$ satisfies the following 'backward master equation' for all configurations $\mathcal{C}$ neither in the set A nor in the set B:

$$\sum_{\mathcal{C}'} W(\mathcal{C} \to \mathcal{C}') E_{B/A}(\mathcal{C}') - W_{\text{out}}(\mathcal{C}) E_{B/A}(\mathcal{C}) = 0, \tag{11}$$

whereas the configurations in the set A or in the set B satisfy the boundary conditions

$$E_{B/A}(\mathcal{C} \in A) = 0, \tag{12}$$

$$E_{B/A}(\mathcal{C} \in B) = 1. \tag{13}$$

The backward master equation (11) does not contain any right-hand side in contrast to equation (3): the iterative elimination of configurations will lead to renormalized transition rates that follows the same two first rules of equation (8).

3.3. Similarities and differences with the strong disorder renormalization of [1, 2]

It turns out that the renormalization rules for the transition rates given in the two first lines of equation (3) are formally identical to the strong disorder rules introduced in [1, 2]. It is thus important to stress here why the interpretation, the goals and the validity of the two approaches are significantly different.

(i) The present renormalization rules are exact for any dynamics defined by a master equation. But they yield results only for observables like first-passage times that satisfy backward master equations with fixed right-hand side.

(ii) In contrast, the strong disorder renormalization procedure introduced in [1, 2] aims to renormalize the forward master equation of equation (1), i.e. the full time evolution of the probability distribution $P_t(\mathcal{C})$. It will become asymptotically exact at large times only for dynamics governed by an 'infinite disorder fixed point' (see more details in [1]). However, whenever it is the case, it can yield results for any universal observable (i.e. exponents or rescaled distributions).

4. Random walk in a two-dimensional self-affine potential

In this section, we apply the method of the previous section to the continuous-time random walk of a particle in a two-dimensional self-affine quenched random potential of Hurst exponent $H = 0.5$. Since we have studied recently in [3] the very same model via some strong disorder renormalization procedure, we refer the reader to [3] and references therein for a detailed presentation of the model and of the numerical method to generate the random potential. Here we simply recall what is necessary for the present approach.

We consider a two-dimensional square lattice of size $L \times L$. The continuous-time random walk in the random potential $U(\vec{r})$ is defined by the master equation

$$\frac{\mathrm{d}P_t(\vec{r})}{\mathrm{d}t} = \sum_{\vec{r}\,'} P_t(\vec{r}\,')W(\vec{r}\,' \to \vec{r}) - P_t(\vec{r})W_{\mathrm{out}}(\vec{r}), \tag{14}$$

where the transition rates are given by the Metropolis choice at temperature T (the numerical data presented below correspond to $T = 1$):

$$W(\vec{r} \to \vec{r}\,') = \delta_{\langle \vec{r},\vec{r}\,'\rangle} \min(1, e^{-(U(\vec{r}\,')-U(\vec{r}))/T}) \tag{15}$$

where the factor $\delta_{\langle \vec{r},\vec{r}\,'\rangle}$ means that the two positions are neighbors on the two-dimensional lattice. The random potential $U(\vec{r})$ is self-affine with Hurst exponent $H = 0.5$:

$$\overline{[U(\vec{r}) - U(\vec{r}\,')]^2} \underset{|\vec{r}-\vec{r}\,'|\to\infty}{\simeq} |\vec{r} - \vec{r}\,'|^{2H}. \tag{16}$$

We focus here on the first-passage time $\tau^{(A)}(\mathcal{C}_0)$ corresponding to the following conditions: (i) the initial configuration $\mathcal{C}_0$ is the center of the square ($x_0 = L/2$, $y_0 = L/2$) and (ii) the set A of 'target configurations' is the set of all boundary sites of the square, i.e. having $x = 1$, $x = L$, $y = 1$ or $y = L$. The first-passage time $\tau^{(A)}(\mathcal{C}_0)$ thus corresponds here to the first exit time t_{exit} from the square $L \times L$ when starting at the center. The appropriate variable is actually the barrier defined as

$$\Gamma_{\mathrm{exit}} \equiv \ln t_{\mathrm{exit}}. \tag{17}$$

In figure 1(*a*), we show the corresponding probability distribution $Q_L(\Gamma_{\mathrm{exit}} \equiv \ln t_{\mathrm{exit}})$ for various sizes $20 \leqslant L \leqslant 80$ with a statistics of $9 \times 10^5 \geqslant n_s(L) \geqslant 36 \times 10^2$ disordered samples.

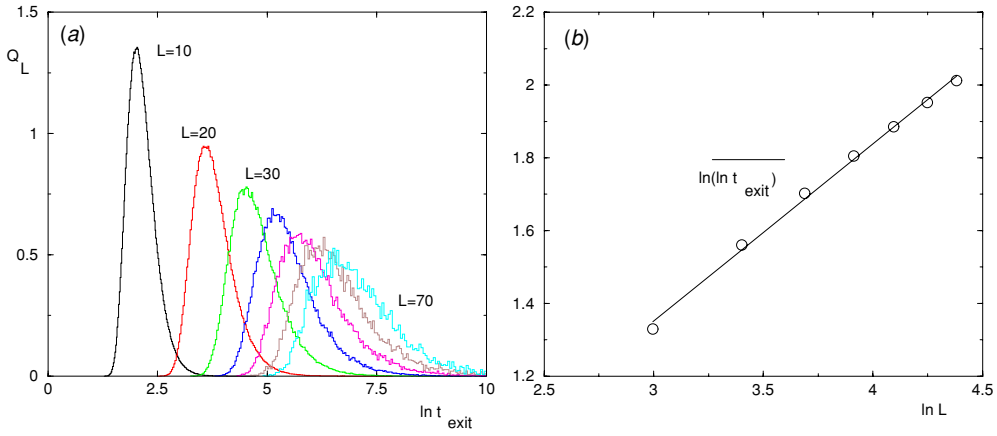

Figure 1. Statistics of the first exit time $t_{\rm exit}$ from a square of size $L \times L$ when starting at the center for the random walk in a self-affine random potential of Hurst exponent $H = 0.5$: (a) probability distribution $Q_L(\Gamma_{\rm exit} = \ln t_{\rm exit})$ for $L = 20, 30, 40, 50, 60, 70$; ($b$) the log–log plot of the disorder-average $\overline{\Gamma_{\rm exit}}(L) = \overline{\ln t_{\rm exit}}(L)$ corresponds to the barrier exponent $\psi = H = 0.5$ (equation (19)).

As shown by the log–log plot of figure 1(b), we find that the disorder-averaged value $\overline{\Gamma}_{\rm exit}(L)$ scales as

$$\overline{\Gamma_{\rm exit}}(L) \underset{L\to\infty}{\propto} L^{\psi} \tag{18}$$

with a barrier exponent ψ of order

$$\psi = H = 0.5. \tag{19}$$

These results are in agreement with scaling arguments on barriers [21, 22], with the strong disorder renormalization approach of [3], and with the computation of the relaxation time to equilibrium [10].

5. Dynamics of the ferromagnetic Sherrington–Kirkpatrick model

As an example of application to a many-body disordered system, we consider in this section the ferromagnetic SK model where a configuration $\mathcal{C} = \{S_i\}$ of N spins $S_i = \pm 1$ has for energy

$$U = -\sum_{1 \leqslant i < j \leqslant N} J_{ij} S_i S_j, \tag{20}$$

where the coupling J_{ij} between two spins S_i and S_j contains a non-random ferromagnetic part J_0 and a random Gaussian part $\tilde{J}_{ij}$ of zero-mean $\overline{\tilde{J}_{ij}} = 0$ and variance unity $\overline{\tilde{J}_{ij}^2} = 1$ with the appropriate mean-field rescalings [23–26]

$$J_{ij} = \frac{J_0}{N-1} + \frac{\tilde{J}_{ij}}{\sqrt{N-1}}. \tag{21}$$

Here we consider the values $J_0 = 2$ and temperature $T = 1$ where the model is in its ferromagnetic phase [23–26] to study its dynamical properties. The Metropolis dynamics

corresponds to the master equation of equation (1) in configuration space with the transition rates

$$W(\mathcal{C} \to \mathcal{C}') = \delta_{\langle \mathcal{C},\mathcal{C}' \rangle} \min(1, e^{-(U(\mathcal{C}')-U(\mathcal{C}))/T}), \tag{22}$$

where the factor $\delta_{\langle \mathcal{C},\mathcal{C}' \rangle}$ means that the two configurations are related by a single spin flip.

We focus here on the first-passage time $\tau^{(A)}(\mathcal{C}_0)$ corresponding to the following conditions: (i) the initial configuration $\mathcal{C}_0$ is the fully ferromagnetic configuration of magnetization $M_N = \sum_{i=1}^N S_i = N$ where all spins are $S_i = +1$ and (ii) the set A of 'target configurations' is the set of all configurations of zero magnetization $M_N = \sum_{i=1}^N S_i = 0$ (we consider only even N). The first-passage time $\tau^{(A)}(\mathcal{C}_0)$ thus corresponds here to the first time $t_{\rm flip}$ where the magnetization M_N vanishes.

We have computed the distribution Q_L of the barrier defined as

$$\Gamma_{\rm flip} \equiv \ln t_{\rm flip} \tag{23}$$

over the disordered samples of even sizes $4 \leqslant N \leqslant 12$ with a statistics of $2 \times 10^8 \geqslant n_s(L) \geqslant 6 \times 10^2$ samples. As a comparison, we have also computed the distribution of the barrier $\Gamma_{eq} \equiv \ln t_{eq}$, where t_{eq} is defined as the largest relaxation time toward equilibrium via the method described in our previous work [10]. Since the system is in its ferromagnetic phase, one expects that the disorder-average of the barrier grows as

$$\overline{\Gamma_{\rm flip}}(N) = \overline{\ln t_{\rm flip}} \underset{N\to\infty}{\propto} N, \tag{24}$$

and this is indeed what we measure both for $\overline{\Gamma_{\rm flip}}(N)$ and for $\overline{\Gamma_{eq}}(N)$ as shown in figure 2(*a*). The width $\Delta(N)$ of the barrier distribution is expected to grow with a subleading exponent $0 < \psi_{\rm width} < 1$:

$$\Delta(N) \equiv \left(\overline{\Gamma^2_{\rm flip}}(N) - (\overline{\Gamma_{\rm flip}}(N))^2\right)^{1/2} \underset{N\to\infty}{\propto} N^{\psi_{\rm width},} \tag{25}$$

but we are not aware of any theoretical prediction or any previous numerical measure of this sample-to-sample fluctuation exponent $\psi_{\rm width}$. This is in contrast with the spin-glass SK model corresponding to $J_0 = 0$, where the barrier exponent has been much studied either theoretically [27, 28] or numerically [10, 29–33].

With our numerical data limited to small sizes $4 \leqslant N \leqslant 12$, we see already the expected linear behavior of the disorder average of equation (24) as shown in figure 2(*a*), but we are unfortunately not able to measure the exponent $\psi_{\rm width}$ of equation (25) from the variance. However, since for these small sizes we can study a large statistics of disordered samples, we have measured the rescaled distribution $\tilde{Q}$ defined as

$$Q_L(\Gamma_{\rm flip}) \sim \frac{1}{\Delta(N)} \tilde{Q}_{\rm flip}\left(u \equiv \frac{\Gamma_{\rm flip} - \overline{\Gamma_{\rm flip}}(N)}{\Delta(N)}\right). \tag{26}$$

We find that the rescaled distribution $\tilde{Q}(u)$ shown in figure 2(*b*) presents at large argument the exponential decay

$$\ln \tilde{Q}_{\rm flip}(u) \underset{u\to+\infty}{\propto} -u^{\eta} \tag{27}$$

with a tail exponent of order

$$\eta \simeq 1.72. \tag{28}$$

We have, moreover, checked that the rescaled distribution $\tilde{Q}_{\rm flip}(u)$ exactly coincides with the rescaled probability distribution $\tilde{Q}_{eq}(u)$ as computed from the method of [10].

To interpret the value of equation (28), one may propose the following rare-event argument. Since the system is in its ferromagnetic phase, it seems natural to expect that the anomalously

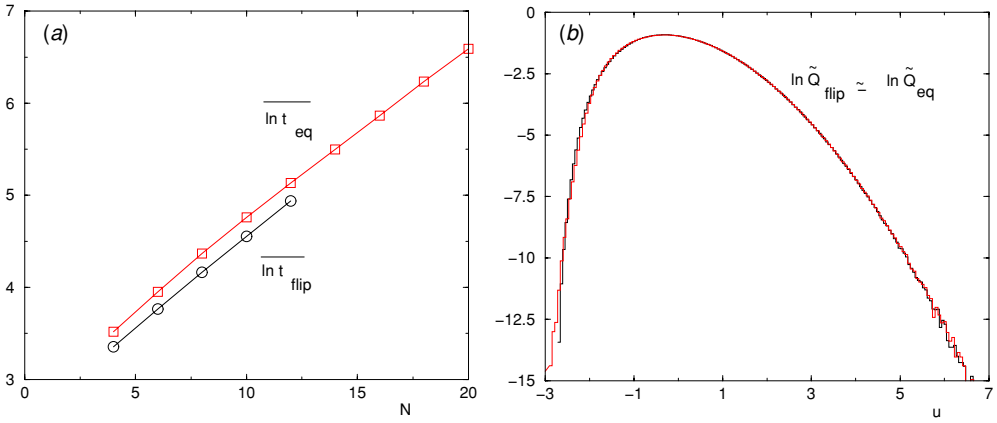

Figure 2. Statistics of the first time $t_{\rm flip}$ where the magnetization vanishes, for the ferromagnetic SK model of N spins (equation (21)): (a) the disorder-average $\overline{\ln t_{\rm flip}}(N)$ grows linearly with N (equation (24)). The disorder-average $\overline{\ln t_{eq}}(N)$ associated with the largest relaxation time $t_{eq}(N)$ toward equilibrium as computed from the method of [10] is also shown for comparison. (b) The rescaled probability distribution $\tilde{Q}_{\rm flip}(u)$ of equation (26), shown here in log scale to see the tail of equation (27), exactly coincides with the rescaled probability distribution $\tilde{Q}_{eq}(u)$ as computed from the method of [10]: the tail exponent is for both of order $\eta \simeq 1.72$ (equation (28)).

large barriers in the dynamics will correspond to the samples that have anomalously strong ferromagnetic contributions coming from the random parts of the couplings in equation (21): with an exponentially rare probability of order $e^{-(cst)N^2}$, the N^2 random variables $\tilde{J}_{ij}$ will be all positive. Then instead of being finite, the local field $h_i = \sum_j J_{ij} S_j$ on spin S_i will be of order $N^{1/2}$, and one thus expects a barrier of order $N^{3/2}$. If one plugs these values in equations (26) and (27), one obtains, for the powers of N in the exponentials, the consistency equation

$$\left(\tfrac{3}{2} - \psi_{\rm width}\right) \eta = 2. \tag{29}$$

For instance $\psi_{\rm width} = 1/2$ would correspond to $\eta = 2$. The value

$$\psi_{\rm width} = \tfrac{1}{3} \tag{30}$$

would correspond to the tail exponent value

$$\eta\left(\psi_{\rm width} = \tfrac{1}{3}\right) = \tfrac{12}{7} = 1.714\ldots, \tag{31}$$

which is extremely close to the value that we measure numerically (equation (28)). A tentative conclusion would thus be the following: at the small sizes that we can study, we cannot measure the width exponent $\psi_{\rm width}$ from the variance, but we can measure the tail exponent η that contains the information on $\psi_{\rm width}$ if one can properly identify the rare events that dominate the tail. In the ferromagnetic phase considered here, we believe that the rare events dominating the tail are the anomalously strong ferromagnetic samples described above, so that our measure of the tail exponent of equation (28) would point toward the value of equation (30) for the width exponent. Of course, this type of indirect reasoning based on rare events remains rather speculative, and a direct measure of $\psi_{\rm width}$ from the variance for large sizes N via Monte Carlo simulations would be very welcome (to the best of our knowledge, the variance has only been measured up to now for the case $J_0 = 0$ in [33]).

We have also tried to apply the same strategy for the SK model for the case $J_0 = 0$, by considering first-passage times defined in terms of overlaps (as was done for instance in the

Monte Carlo simulations of [32]), but our numerical results have turned out to present too big finite-size effects for the sizes that we were able to study. We thus refer the interested reader to our previous work based on another method [10], where the histograms of the relaxation time for the case $J_0 = 0$ have been analyzed.

6. Conclusion

To avoid the simulation of the dynamics of disordered systems which can be extremely slow, we have proposed in this paper to focus on first-passage times that satisfy 'backward master equations'. We have shown that these equations satisfy exact renormalization rules upon the iterative elimination of configurations. We have explained the similarities and differences with the strong disorder renormalization of [1, 2]. We have then tested numerically this approach for two types of disordered models: (i) for the random walk in a two-dimensional self-affine random potential of Hurst exponent $H = 1/2$, we have computed the statistics of the first exit time from a square of size $L \times L$ if one starts at the square center and (ii) for the dynamics of the ferromagnetic SK model, we have studied the statistics of the first passage time t_f to zero-magnetization when starting from a fully magnetized configuration. We have compared with the results concerning the largest relaxation time toward equilibrium obtained with the method of [10]. Our conclusion is that the first-passage method is reliable to measure dynamical properties of disordered systems. Although in some cases, it takes more CPU time than the method of [10], it can have several advantages in other cases.

(i) It does not require the detailed balance condition (in contrast to [10]).
(ii) The CPU time depends only on the size of configuration space, but not at all on the disorder realization and on the time scales involved that can be arbitrarily large (in contrast to [10] where the convergence of the iteration method depends on the disorder sample and on the temperature).
(iii) The freedom in the choice of the initial condition and of the 'target configurations', can be useful to study the time scales associated with various dynamical processes (whereas the method of [10] focuses on the largest relaxation time toward equilibrium).

From a numerical point of view, it is clear that both the present method and the method of [10] require to work in the entire configuration space, whose dimension grows exponentially with the number of degrees of freedom. As a consequence, these methods allow us to obtain results for small sizes but with a large statistics of samples: it turns out that numerically, the convergence toward a fixed rescaled distribution is usually very rapid (as can be checked in each case), so that the tails exponents of these distributions can be measured to obtain information on typical exponents via matching arguments between typical and rare events (see also [34] and references therein for the application of the same type of arguments to the distribution of the ground state energy of some disordered systems). These methods are thus complementary to the usual Monte Carlo dynamical simulations of disordered systems that follow the opposite strategy: their aim is to study only the averaged value for the biggest sizes available (with necessarily smaller statistics).

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